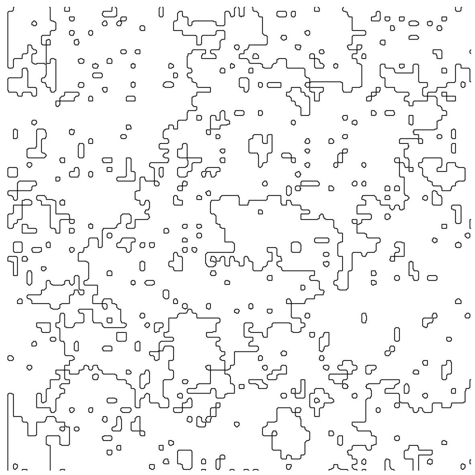
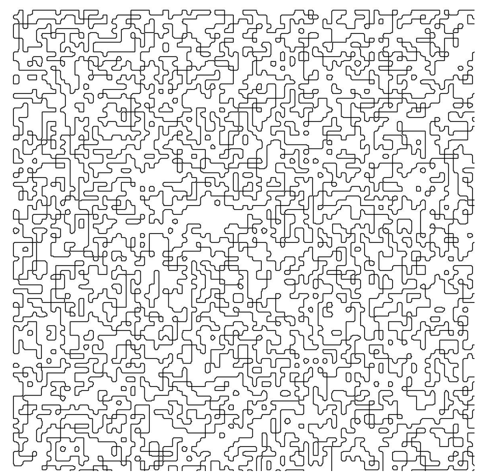
(1) [3 pt] These are two simulations of the low-temperature representation of the Ising model, one where $\beta$ is small and one where $\beta$ is large. Which one is more probable to be which?





(2) [3 pt] Show that for the Ising model with + boundary conditions on a finite connected graph, and any vertices $a, b$ and $c$, we have $\mathbb{E}\left[\sigma_a \sigma_b \sigma_c\right] > 0$.

(3) [4 pt] Consider the subgraph $\mathbb{G} = \{(x,y)\,|1 \le x \le 2020, 1 \le y \le 2020\}$ of $\mathbb{Z}^2$ and $\tau$ be a dimer tilling of $\mathbb{G}$ picked uniformly at random. What is the probability that the edge $\langle (1,1),(1,2) \rangle$ belongs to $\tau$?

(4) [3+3 pt] Consider the subgraph $\mathbb{G} = \{(x, y) \,|1 \leq x \leq 2019, 1 \leq y \leq 2020\}$ of $\mathbb{Z}^2$.

(a) In a dimer tilling of $\mathbb{G}$ picked uniformly at random, the probability that the right-most column is occupied by 1010 adjacent vertical dimers is greater than

$$\sqrt{\frac{1}{\prod_{k=1}^{2019} \prod_{\ell=1}^{2020} \left(2\cos\left(\frac{\pi k}{2020}\right) + 2i\cos\left(\frac{\pi \ell}{2021}\right)\right)}}.$$

True of False ? Justify in either case !

(b) The probability that all the edges

$$\{\langle (1,1)\,, (1,2)\rangle\,, \langle (2,2)\,; (2,3)\rangle\,, \langle (3,3)\,; (3,4)\rangle\,, \ldots, \langle (2019, 2019)\,; (2019, 2020)\rangle\}$$

belong to a a dimer tilling of $\mathbb{G}$ picked uniformly at random is greater than

$$\sqrt{\frac{1}{\prod_{k=1}^{2019} \prod_{\ell=1}^{2020} \left(2\cos\left(\frac{\pi k}{2020}\right) + 2i\cos\left(\frac{\pi \ell}{2021}\right)\right)}}.$$

True of False ? Justify in either case !

(5) [6 pt] Let $R = [a,b] \times [c,d]$ be a rectangular box. Discretize $R$ by a fine hexagonal lattice of mesh size $\delta > 0$ and consider the usual critical percolation (color independently the faces in black/white with probability $\frac{1}{2}/\frac{1}{2}$). Let $A$ be the event that there is a black cluster linking the left side to the right side of the box. Let $B$ be the event that there is a black cluster touching all four sides of the rectangular box.
Show that:

$$\mathbb{P}\left(B\right) \geq \mathbb{P}(A)\left(1 - \mathbb{P}(A)\right).$$

(6) [3+3 pt] Let $\mathbb{G}$ be a connected finite graph with boundary $\partial\mathbb{G}$. Consider a simple random walk $(X_n)_{n\geq 0}$ on $\mathbb{G}$ starting from $x_0 \in \mathbb{G} \cup \partial\mathbb{G}$. Consider $\tau_{\partial\mathbb{G}}(x_0) = \inf\{n \geq 0 : X_n \in \partial\mathbb{G}\}$.

(a) The following equality holds:

$$\mathbb{E}\left[\tau_{\partial\mathbb{G}}(x_0)\right] = \sum_{y \in \mathbb{G}\cup\partial\mathbb{G}} G(y, x_0),$$

where $G$ is the Green's function of the simple random walk $(X_n)_{n\geq 0}$. True or False ? Justify in either case !

(b) Show that $\mathbb{E}[\tau_{\partial\mathbb{G}}]$ is the unique solution of

$$\begin{cases} \Delta\mathbb{E}[\tau_{\partial\mathbb{G}}] = -1 & \text{on } \mathbb{G}, \\ \mathbb{E}[\tau_{\partial\mathbb{G}}] = 0 & \text{on } \partial\mathbb{G}. \end{cases}$$

where $\Delta f(x) = \frac{1}{\#\{y, y\leftrightarrow x\}} \sum_{y\leftrightarrow x} [f(y) - f(x)]$ and the sum is over the neighbours of $x$.

(7) [9 pt] Consider a simple random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ starting from 1. Let $\tau_0 = \inf\{n \geq 0 : X_n = 0\}$.

(a) Show that $\mathbb{P}\{\tau_0 < +\infty\} = 1$.

(b) For any integer $N > 0$, compute the Green's function of the simple random walk $(X_n)_{n\geq 0}$ on $\{0, \ldots, N\}$ (with boundary $\{0, N\}$).

(c) *Little bit harder:* Show that $\mathbb{E}[\tau_0] = +\infty$ *(Hint: study* $\mathbb{E}[\tau_{0,N}]$ *where* $\tau_{0,N} = \inf\{n \geq 0 : X_n \in \{0, N\}\}$*).*

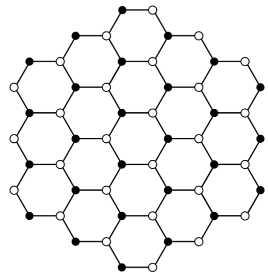
(8) [6 pt] Using Wilson's algorithm, explain why for any finite connected graph $G$, the law of the edges visited by a loop-erased random walk from $x$ to $y$ is the same as the law of the edges visited by a loop-erased random walk from $y$ to $x$.

(9) [6 pt] Let $\ell = \{z \in \mathbb{C} : \mathfrak{Re}(z) = 0\}$ denote the imaginary line. Consider a Jordan domain $(\Omega, n, w, s, e)$, with $n, w, s, e \in \partial\Omega$ in counterclockwise order, with $n, s \in \ell$ and such that $\Omega$ is symmetric with respect to $\ell$ and that $w$ is is the symmetric of $e$ with respect to $\ell$.

(a) Make a picture.

(b) Let $\Delta$ denote the equilateral triangle with vertices $1, \frac{\sqrt{3}}{3}i, -\frac{\sqrt{3}}{3}i$ and let $\varphi : \Omega \to \Delta$ be the conformal mapping such that $\varphi(e) = 1, \varphi(n) = \frac{\sqrt{3}}{3}i, \varphi(w) = -\frac{\sqrt{3}}{3}i$. Using a percolation argument, show that $\varphi(s) = \frac{1 - \frac{\sqrt{3}}{3}i}{2}$.

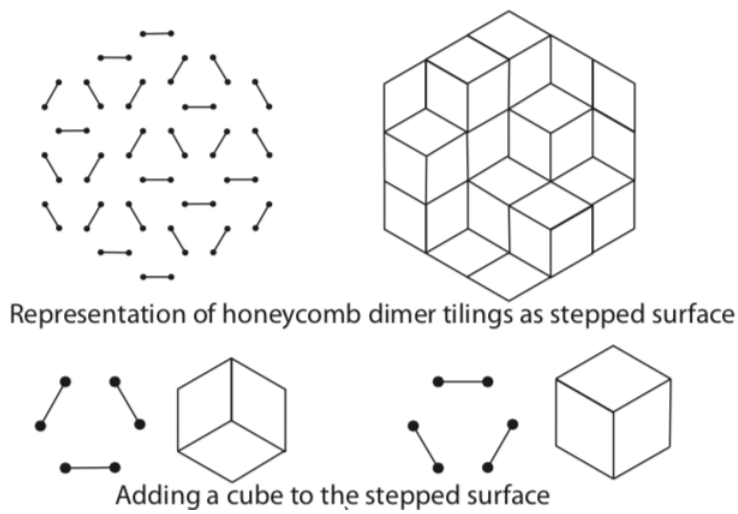
(10) [9 pt] Dimers on hexagonal grid. We consider the following hexagonal grid:



and its reduced adjacency matrix given by

$$A_{i,j} = \delta_{b_i \sim w_j},$$

where $b_1, \ldots, b_n$ are the black vertices and $w_1, \ldots, w_n$ are the white vertices. Show that the number of tillings on this hexagonal grid is given by $|\det(A)|$. *Hint: consider the following 3D representation of dimers on hexagonal grid.*



(11) [12 pt] Explain why the 3D Ising model on the cubic box $\{0,\dots,n\}\times\{0,\dots,n\}\times\{0,\dots,n\}$ with $\beta \to +\infty$ with $+$ boundary conditions on the three faces such that either $x=0$, $y=0$ or $z=0$ and $-$ boundary conditions on the three faces such either $x=n$, $y=n$ or $z=n$, corresponds to a dimer model on the hexagonal domain shown in the previous exercise.

(a) *Harder*: if we define a Markov chain dynamics, on the hexagon dimers by randomly performing moves as above (choosing them uniformly among the possible elementary moves available), do we converge to the uniform distribution on dimers? *Hint: look at the equilibrium measure.*

(b) *Much harder*: what is the difference between this dynamics and the Metropolis dynamics for a corresponding 3D Ising model with $\beta = +\infty$? Why does one give a uniform distribution on dimer configurations and the other one does not?

(12) [1 pt] What was your favorite topic covered in class?