We continue the study of the face percolation on hexagonal graph for $p = \frac{1}{2}$ (it is the last exercise sheet on this topic).

Exercise 1. *Convergence of* $\mathcal{H}_\delta^1 + \mathcal{H}_\delta^2 + \mathcal{H}_\delta^3$

In the lecture we saw the definition of the percolation observables $\mathcal{H}_\delta^1, \mathcal{H}_\delta^2, \mathcal{H}_\delta^3$ and the proof of convergence of $\mathcal{H}_\delta^1 + \frac{\sqrt{3}}{3} i\mathcal{H}_\delta^2 - \frac{\sqrt{3}}{3} i\mathcal{H}_\delta^3$. In this exercise, we will prove the convergence of $\mathcal{H}_\delta^1 + \mathcal{H}_\delta^2 + \mathcal{H}_\delta^3$ using similar arguments. Recall that Ω is a bounded simply connected domain with piecewise smooth boundary which is marked at distinct counterclockwise points $a_1,\ a_2,\ a_3$.

(1) Show that one can extract a sub-sequence of δ_n such that $\mathcal{H}_{\delta_n}^1 + \mathcal{H}_{\delta_n}^2 + \mathcal{H}_{\delta_n}^3$ converges as $n \to \infty$.

Solution. We have seen that the functions $\mathcal{H}_\delta^1$ are Hölder continuous uniformly in δ. Thus, $\mathcal{H}_\delta^1 + \mathcal{H}_\delta^2 + \mathcal{H}_\delta^3$ is also Hölder continuous uniformly in δ. Using the results of last week (Arzela-Ascoli) we can extract a subsequence $\delta_n \to 0$ such that $\mathcal{H}_{\delta_n}^1 + \mathcal{H}_{\delta_n}^2 + \mathcal{H}_{\delta_n}^3$ converges as $n \to \infty$.

(2) Show that the limiting function is holomorphic.

Hint: since the proof follows the one explained in the lesson, what is expected is that you give the only thing which differs from the proof given in the lesson.

Solution. In order to prove that the limiting function f is holomorphic, we use the arguments seen in the lesson :

(a) Let f denote the limit of the converging subsequence. We want to prove that $\oint_\gamma f = 0$ for any closed curve γ. (Morera's theorem)

(b) For $\mathcal{H}_\delta^1 + \mathcal{H}_\delta^2 + \mathcal{H}_\delta^3$ we study the discrete sum over oriented edge configurations and show this converges to 0.

(c) We do the same computation as in the lecture except that we consider $\mathcal{H}_{\delta_n}^1 + \mathcal{H}_{\delta_n}^2 + \mathcal{H}_{\delta_n}^3$ and not $\mathcal{H}_{\delta_n}^1 + \tau\mathcal{H}_{\delta_n}^2 + \tau^2\mathcal{H}_{\delta_n}^3$.

(d) Thus, we get

$$\oint_\gamma f \sim \sum_{\vec{e}} \partial_{\vec{e}}^+ \mathcal{H}_\delta^1 \overrightarrow{e^*} + \sum_{\vec{e}} \partial_{\tau\vec{e}}^+ \mathcal{H}_\delta^2 \tau \overrightarrow{e^*} + \sum \partial_{\tau^2\vec{e}}^+ \mathcal{H}_\delta^3 \tau^2 \overrightarrow{e^*} + O(\delta).$$

After using the Cauchy Riemann equations we get :

$$\sum_{\vec{e}} \partial_{\vec{e}}^+ \mathcal{H}_\delta^1 \overrightarrow{e^*} + \sum_{\vec{e}} \partial_{\vec{e}}^+ \mathcal{H}_\delta^1 \tau \overrightarrow{e^*} + \sum \partial_{\vec{e}}^+ \mathcal{H}_\delta^1 \tau^2 \overrightarrow{e^*} + O(\delta) = \sum_{\vec{e}} \overrightarrow{e^*} \partial_{\vec{e}}^+ \mathcal{H}_\delta^1 \left(1 + \tau + \tau^2\right) + O(\delta)$$

and since $1 + \tau + \tau^2 = 0$ we obtain the desired result.

(3) Show that $\mathcal{H}_\delta^1 + \mathcal{H}_\delta^2 + \mathcal{H}_\delta^3$ converges to 1 on $\overline{\Omega}$ as δ goes to 0.

Solution. The limiting function f is

(a) holomorphic

(b) purely real.

Yet any holomorphic function which is purely real is constant. Indeed, using Cauchy-Riemann equations:

$$\begin{aligned} \partial_x \Re f &= \partial_y \Im f = 0, \\ \partial_y \Re f &= -\partial_x \Im f = 0 \end{aligned}$$

Thus $\partial_x \Re f = \partial_y \Re f = 0$: f is constant. We need to understand which constant it is: using the boundary study we have done during the lesson, we get that $\lim_{\delta\to 0} \mathcal{H}_\delta^1(a_1) = 1$ and $\lim_{\delta\to 0} \mathcal{H}_\delta^2(a_1) + \mathcal{H}_\delta^3(a_1) = 0$, hence $\lim_{\delta\to\infty} \mathcal{H}_\delta^1(a_1) + \mathcal{H}_\delta^2(a_1) + \mathcal{H}_\delta^3(a_1) = 1$. Thus the function $\mathcal{H}_\delta^1 + \mathcal{H}_\delta^2 + \mathcal{H}_\delta^3$ converges to 1 on $\overline{\Omega}$ as δ goes to 0.

Exercise 2. *Hitting distribution*

Consider the equilateral triangle $T \subset \mathbb{C}$ whose vertices are $0, e^{\pm\pi i/6}$. Get the hexagonal discretisation T_δ of T as usual, and consider the critical face percolation on T_δ.

Let us color the complement of the triangle in the right half plane $\{\Re z > 0\} \setminus T_\delta$ black on the top side (positive imaginary part) and white on the bottom (negative imaginary part). Each site percolation configuration gives us an interface: the well-defined path defining the interface between the black coloring on the upper half plane and the white coloring on the lower half plane (there can also be some islands of each color above and below this path). We would like to study the hitting distribution of this path on the right side $\left[e^{\pi i/6}, e^{-\pi i/6}\right]$: where does the interface end? In terms of the percolation on T_δ, this corresponds to the distribution of the highest white face $W \in \left[e^{\pi i/6}, e^{-\pi i/6}\right]$ connected to the bottom $\left[0, e^{-\pi i/6}\right]$ (if there is no such face, we set its location as $e^{-\pi i/6}$).

(1) Recall Cardy's theorem for the limit of the crossing probability in a general bounded simply connected domain Ω.

Solution. It is in your lesson.

(2) Using Cardy's formula, show that $\lim_{\delta\to 0} \mathbb{P}_{T_\delta}\left[\Im(W) > h\right] = \frac{1}{2} - h$ for $h \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Conclude that W converges in distribution to the uniform variable on $\left[e^{-i\pi/6}, e^{i\pi/6}\right]$ as $\delta \to 0$.

Solution. Let us remark that :

$$\lim_{\delta\to 0} \mathbb{P}_{T_\delta}\left[\Im(W) > h\right] = \lim_{\delta\to 0} \mathbb{P}\left[\exists \text{ white crossing from } \left[\frac{\sqrt{3}}{2} + ih, e^{i\pi/6}\right] \text{ to } \left[0, e^{-i\pi/6}\right]\right]$$

Thus by Cardy's formula, we have that :

$$\lim_{\delta\to 0} \mathbb{P}_{T_\delta}\left[\Im(W) > h\right] = \frac{|e^{i\pi/6} - \left(\frac{\sqrt{3}}{2} + ih\right)|}{1} = \frac{1}{2} - h.$$

This implies that $\Im(W)$ is uniform in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and thus, W is uniform on $\left[e^{-i\pi/6}, e^{i\pi/6}\right]$ as $\delta \to 0$.

(3) Now consider the deformed triangle made out of two straight line segments from 0 to $e^{\pm\pi i/3}$ and a third segment connecting $e^{\pm\pi i/3}$ through the parabola $y^2 = \frac{9}{4} - 3x$ between $y = \pm\frac{\sqrt{3}}{2}$. What is the hitting distribution in this case? *Hint: The conformal map from this domain to the equilateral triangle T is very simple!*

Solution. Let us call $\tilde{T}$ the deformed triangle. Let us remark that

$$\begin{aligned} \psi : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\rightarrow z^2 \end{aligned}$$

sends T onto $\tilde{T}$. Indeed, it is easy to see that the segments $\left[0, e^{i\pi/6}\right]$ and $\left[0, e^{-i\pi/6}\right]$ are sent on $\left[0, e^{i\pi/3}\right]$ and $\left[0, e^{-i\pi/3}\right]$. Also if $z = \frac{\sqrt{3}}{2} + ih$ then $\psi(z) = \frac{3}{4} - h^2 + \sqrt{3}hi$. Thus

$$\Im(\psi(z))^2 = 3h^2 = \frac{9}{4} - 3\Re(\psi(z))$$

which shows that the segment $\left[e^{-i\pi/6}, e^{i\pi/6}\right]$ is sent onto the segment of parabola $y^2 = \frac{9}{4} - 3x$ between $y = \pm\frac{\sqrt{3}}{2}$.

Now, let us remark that Cardy's formula tells us that the crossing probability is a conformal invariant. This implies that the law of the hitting distribution is also conformally invariant, since

$$\lim_{\delta\to 0} \mathbb{P}\left(\Im\left(\tilde{W}\right) > \tilde{h}\right) = \lim_{\delta\to 0} \mathbb{P}\left(\exists \text{ white crossing from } \left[\left(\frac{1}{3}\left(\frac{9}{4} - \tilde{h}^2\right) + i\tilde{h}\right), e^{i\pi/3}\right]_{\tilde{T}} \text{ to } \left[0, e^{-i\pi/3}\right]\right)$$

and we can then apply Cardy's theorem. Let us denote by $\tilde{W}$ the hitting point for the deformed triangle $\tilde{T}$: the conformal invariance tells us that $\tilde{W}$ and $\psi(W)$ have the same law. Thus for $\tilde{h}$ in the segment of $\tilde{T}$ connecting $e^{\pm\pi i/3}$,

$$\lim_{\delta\to 0} \mathbb{P}\left(\Im\left(\tilde{W}\right) > \tilde{h}\right) = \lim_{\delta\to 0} \mathbb{P}\left(\Im\left(\psi(W)\right) > \Im\left(\psi(\sqrt{3}/2 + ih)\right)\right)$$

where we define h by the relation $\tilde{h} = \Im\left(\psi(\sqrt{3}/2 + ih)\right)$ and thus :

$$\begin{aligned} \lim_{\delta\to 0} \mathbb{P}\left(\Im\left(\tilde{W}\right) > \tilde{h}\right) &= \lim_{\delta\to 0} \mathbb{P}\left(\Im\left(\psi(W)\right) > \Im\left(\psi(\sqrt{3}/2 + ih)\right)\right) \\ &= \lim_{\delta\to 0} \mathbb{P}\left(\Im(W) > \Im\left(\sqrt{3}/2 + ih\right) = h\right) \\ &= \frac{1}{2} - h \end{aligned}$$

$\tilde{h} = \Im\left(\psi(\sqrt{3}/2 + ih)\right)$ implies that $h = \frac{\tilde{h}}{\sqrt{3}}$. This implies that

$$\lim_{\delta \to 0} \mathbb{P}\left(\Im\left(\tilde{W}\right) > \tilde{h}\right) = \frac{1}{2} - \frac{1}{\sqrt{3}}\tilde{h}.$$

This implies that $\Im\left(\tilde{W}\right)$ is uniform between $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$, and thus

$$\tilde{W} \sim \frac{1}{3}\left(\frac{9}{4} - U^2\right) + iU$$

where $U \sim Unif\left(\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]\right)$.