We continue the study of the face percolation on hexagonal graph for $p = \frac{1}{2}$ (it is the last exercise sheet on this topic).

Exercise 1. *Convergence of* $\mathcal{H}_\delta^1 + \mathcal{H}_\delta^2 + \mathcal{H}_\delta^3$

In the lecture we saw the definition of the percolation observables $\mathcal{H}_\delta^1, \mathcal{H}_\delta^2, \mathcal{H}_\delta^3$ and the proof of convergence of $\mathcal{H}_\delta^1 + \frac{\sqrt{3}}{3} i\mathcal{H}_\delta^2 - \frac{\sqrt{3}}{3} i\mathcal{H}_\delta^3$. In this exercise, we will prove the convergence of $\mathcal{H}_\delta^1 + \mathcal{H}_\delta^2 + \mathcal{H}_\delta^3$ using similar arguments. Recall that Ω is a bounded simply connected domain with piecewise smooth boundary which is marked at distinct counterclockwise points $a_1,\ a_2,\ a_3$.

(1) Show that one can extract a sub-sequence of δ_n such that $\mathcal{H}_{\delta_n}^1 + \mathcal{H}_{\delta_n}^2 + \mathcal{H}_{\delta_n}^3$ converges as $n \to \infty$.
(2) Show that the limiting function is holomorphic.
Hint: since the proof follows the one explained in the lesson, what is expected is that you give the only thing which differs from the proof given in the lesson.
(3) Show that $\mathcal{H}_\delta^1 + \mathcal{H}_\delta^2 + \mathcal{H}_\delta^3$ converges to 1 on $\overline{\Omega}$ as δ goes to 0.

Exercise 2. *Hitting distribution*

Consider the equilateral triangle $T \subset \mathbb{C}$ whose vertices are $0, e^{\pm \pi i/6}$. Get the hexagonal discretisation T_δ of T as usual, and consider the critical face percolation on T_δ.

Let us color the complement of the triangle in the right half plane $\{\Re z > 0\} \setminus T_\delta$ black on the top side (positive imaginary part) and white on the bottom (negative imaginary part). Each site percolation configuration gives us an interface: the well-defined path defining the interface between the black coloring on the upper half plane and the white coloring on the lower half plane (there can also be some islands of each color above and below this path). We would like to study the hitting distribution of this path on the right side $\left[e^{\pi i/6}, e^{-\pi i/6}\right]$: where does the interface end? In terms of the percolation on T_δ, this corresponds to the distribution of the highest white face $W \in \left[e^{\pi i/6}, e^{-\pi i/6}\right]$ connected to the bottom $\left[0, e^{-\pi i/6}\right]$ (if there is no such face, we set its location as $e^{-\pi i/6}$).

(1) Recall Cardy's theorem for the limit of the crossing probability in a general bounded simply connected domain Ω.
(2) Using Cardy's formula, show that $\lim_{\delta\to 0} \mathbb{P}_{T_\delta}\left[\Im(W) > h\right] = \frac{1}{2} - h$ for $h \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Conclude that W converges in distribution to the uniform variable on $\left[e^{-i\pi/6}, e^{i\pi/6}\right]$ as $\delta \to 0$.
(3) Now consider the deformed triangle made out of two straight line segments from 0 to $e^{\pm\pi i/3}$ and a third segment connecting $e^{\pm \pi i/3}$ through the parabola $y^2 = \frac{9}{4} - 3x$ between $y = \pm\frac{\sqrt{3}}{2}$. What is the hitting distribution in this case? *Hint: The conformal map from this domain to the equilateral triangle T is very simple!*