Recall that a simple random walk on a graph is called *recurrent* if it returns to the starting point with probability 1, and *transient* otherwise.

Exercise 1. *General knowledge on the recurrence/transience of random walks*

Let G be a general graph (locally finite), let v be a vertex of G and $(S_n)_{n\geq 0}$ be a simple random walk starting at v. We denote by $\mathbb{P}_v$ the corresponding probability measure.

1. Explain what a simple random walk on G is.
2. Prove that $(S_n)_{n\geq 0}$ is recurrent if and only if
$$\sum_{n=0}^{\infty} \mathbb{P}_v \left(S_n = v\right) = \infty.$$
3. Let us suppose that G is connected and v, w are vertices of G.
   (a) Show that a simple random walk on G is recurrent when started from v if and only if it is recurrent when started from w.
   (b) Show that if a simple random walk $(S_n)_{n\geq 0}$ on G is recurrent when started from v, then for any vertex w of G, $\mathbb{P}_v \left(\exists n,\, S_n = w\right) = 1$ and $\mathbb{P}_w \left(\exists n,\, \tilde{S_n} = v\right) = 1$ where $(\tilde{S_n})_{n\geq 0}$ is a simple random walk on G starting at w.
4. Show that a simple random walk on a finite graph is recurrent.
5. Show that a simple random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$ is recurrent if and only if
$$\sum_{n=0}^{\infty} \mathbb{P}_{\vec{0}} \left(S_{2n} = \vec{0}\right) = \infty.$$

Exercise 2. *Universality of the recurrence for random walks on* $\mathbb{Z}$

Consider a random walk on $\mathbb{Z}$ defined using identically independent jumps : $S_n = Z_1 + \cdots + Z_n$ (Z_i are i.i.d. $\mathbb{Z}$-valued random variables). Let us suppose that Z_1 satisfies $\mathbb{E}\left(|Z_1|\right) < \infty$.

1. Prove that if $\mathbb{E}\left(Z_1\right) \neq 0$ then S_n is transient.
2. What is the derivative of $\phi(t) = \mathbb{E}\left(e^{itZ_1}\right)$ at 0 ? Give the Taylor expansion of $\phi(t)$ at 0 at order 2.
3. Using the previous point, prove that if Z_1 is symmetric ($-Z_1$ has the same law as Z_1) then S_n is recurrent. *Hint: use the derivation using the Fourier transform as seen in the lesson.*