Let $G \subseteq \mathbb{Z}^d$ be a finite graph with boundary ∂G (i.e. the vertices in $\mathbb{Z}^d$ having one neighbor in G). The Laplacian on G with boundary conditions b is the linear operator $\Delta_G^b : \mathbb{R}^G \to \mathbb{R}^G$ defined on any function $f : G \to \mathbb{R}$ by the relation

$$\Delta f(x) = \frac{1}{2d} \sum_{y \sim x} [f(y) - f(x)],$$

where $x \in G$, $f(y) = b(y)$ for each $y \in \partial G$ and $y \sim x$ means that $|y - x| = 1$.

The *differential* df of a function f on $G \sqcup \partial G$ is a function defined on any oriented edge (x, y) of $G \sqcup \partial G$ as

$$df(x, y) := f(y) - f(x).$$

The scalar products on the set of functions $\mathbb{R}^G$ is defined as

$$\langle f, g \rangle_{\mathbb{R}^G} := \sum_{x \in G} f(x) g(x).$$

Similarly, if $\vec{\mathcal{E}}$ is a set of oriented edges in G, we define the scalar product on $\mathbb{R}^{\vec{\mathcal{E}}}$ as

$$\langle f, g \rangle_{\mathbb{R}^{\vec{\mathcal{E}}}} := \sum_{(x,y) \in \vec{\mathcal{E}}} f(x, y) g(x, y).$$

Exercise 1. *General knowledge.* Let G be a finite subgraph of $\mathbb{Z}^d$ with boundary ∂G.

(1) Show that $\Delta = \Delta_G^0$ is negative definite. *Hint: Show that* $\langle \Delta f, g \rangle_{\mathbb{R}^G} = -\frac{1}{4d} \langle df, dg \rangle_{\mathbb{R}^{\vec{\mathcal{E}}(G)}} - \frac{1}{2d} \langle df, dg \rangle_{\mathbb{R}^{\vec{\mathcal{E}}_{out}(\partial G)}}$, *where* $\vec{\mathcal{E}}(G)$ *is the set of oriented edges in* G *and* $\vec{\mathcal{E}_{out}}(\partial G)$ *is the set of oriented edges of the form* (x, y) *where* $x \in G$ $y \in \partial G$.

Solution. We can make the following derivation:

$$\begin{aligned}
\langle \Delta f, g \rangle_{\mathbb{R}^G} &= \frac{1}{2d} \sum_{x \in G} \sum_{y \sim x} (f(y) - f(x)) g(x) \\
&= -\frac{1}{2d} \sum_{(x,y) \in \vec{\mathcal{E}}(G) \sqcup \vec{\mathcal{E}}_{out}(\partial G)} (f(y) - f(x)) (g(y) - g(x)) + \frac{1}{2d} \sum_{(x,y) \in \vec{\mathcal{E}}(G) \sqcup \vec{\mathcal{E}}_{out}(\partial G)} (f(y) - f(x)) g(y) \\
&= -\frac{1}{2d} \langle df, dg \rangle_{\mathbb{R}^{\mathcal{E}(\vec{G})}} - \frac{1}{2d} \langle df, dg \rangle_{\mathbb{R}^{\vec{\mathcal{E}}_{out}(\partial G)}} + \frac{1}{2d} \sum_{(x,y) \in \vec{\mathcal{E}}(G)} (f(y) - f(x)) g(y).
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\frac{1}{2d} \sum_{(x,y) \in \vec{\mathcal{E}}(G)} (f(y) - f(x)) g(y) &= \frac{1}{4d} \sum_{(x,y) \in \vec{\mathcal{E}}(G)} (f(y) - f(x)) g(y) + \sum_{(x,y) \in \vec{\mathcal{E}}(G)} (f(x) - f(y)) g(x) \\
&= \frac{1}{4d} \sum_{(x,y) \in \vec{\mathcal{E}}(G)} (f(y) - f(x)) (g(y) - g(x)) \\
&= \frac{1}{4d} \langle df, dg \rangle_{\mathbb{R}^{\vec{\mathcal{E}}(G)}},
\end{aligned}$$

hence $\frac{1}{2d} \sum_{(x,y) \in \vec{\mathcal{E}}(G)} (f(y) - f(x)) g(y) = \frac{1}{4d} \langle df, dg \rangle_{\mathbb{R}^{\vec{\mathcal{E}}(G)}}$ which, when substituted into the previous expression, indeed gives

$$\langle \Delta f, g \rangle_{\mathbb{R}^G} = -\frac{1}{4d} \langle df, dg \rangle_{\mathbb{R}^{\vec{\mathcal{E}}(G)}} - \frac{1}{2d} \langle df, dg \rangle_{\mathbb{R}^{\vec{\mathcal{E}}_{out}(\partial G)}}.$$

(2) Let ψ be an harmonic function such that ψ is null on ∂G. Show that ψ is null on G. Let h_1 and h_2 be two functions respectively on G and ∂G. Let f, g be solutions of

$$\begin{cases} \Delta \phi = h_1 & \text{in } G, \\ \phi = h_2 & \text{on } \partial G \end{cases}$$

show that $f = g$.

Solution. The solution follows from a "maximum principle-like" argument. More precisely, if ψ is an harmonic function such that ψ is null on ∂G, then $|\psi|$ satisfies :

$$|\psi(x)| = \frac{1}{2d}\left|\sum_{y\sim x}\psi(y)\right| \leq \frac{1}{2d}\sum_{y\sim x}|\psi(y)|.$$

Since $G \sqcup \partial G$ is finite, there exists x_0 such that $|\psi|(x_0) = \max_{x\in G}|\psi|(x)$. If $x_0 \in \partial G$ we can conclude. If $x_0 \in G$, then :

$$|\psi(x_0)| \leq \frac{1}{2d}\sum_{y\sim x_0}|\psi(y)| \leq |\psi(x_0)|.$$

This implies that for any $y \sim x$, $|\psi(y)| = |\psi(x_0)| = \max_G |\psi|(x)$. The function takes the same value on the neighbours of x and thus it must take the same value on $G \sqcup \partial G$: we can again conclude. The fact that $f = g$ follows from the previous argument using $\psi = f - g$.

(3) Let $B \subset \partial G$. Recall the two definitions of the harmonic measure $H_B(\cdot)$ defined on G.

Solution. The definitions are :

(a) The unique solution to the PDE problem :

$$\begin{cases} \Delta H_B(x) = 0 & \text{in } G, \\ H_B(x) = 1 & \text{on } B, \\ H_B(x) = 0 & \text{on } \partial G \setminus B. \end{cases}$$

(b) The function :

$$H_B(x) = \mathbb{P}^x\left[S_{\tau_{\partial G}} \in B\right],$$

where $\tau_{\partial G}$ is the first time the random walk $(S_n)_{n\geq 0}$ visits ∂G.

Exercise 2. *Green function*

Let us consider A a finite subgraph of $\mathbb{Z}^d$, and let ∂A be the set of points in $\mathbb{Z}^d \setminus A$ that are adjacent to a point in A. Let $(S_n)_{n\geq 0}$ be a random walk started from a point in A. We define $\tau_A = \min\{n \geq 0, S_n \notin A\}$ which is the first time the random walk leaves A.

Recall that the *Green's function* G_A *defined on* $\mathbb{Z}^d \times \mathbb{Z}^d$ *is*

$$G_A(x,y) = \mathbb{E}^x\left[\#\{0 \leq n < \tau_A | S_n = y\}\right]$$

where we recall that $\mathbb{P}^x$ refers to the probability law of a simple random walk started at x and $\mathbb{E}^x$ is the associated expectation.

(1) Prove that

$$\tau_A < \infty, \qquad \mathbb{E}(\tau_A) < \infty, \qquad G_A(x,y) < \infty$$

Remark. Be careful: for $d \leq 2$ and $A \subset \mathbb{Z}^d$ not necessarily finite it still holds that $\tau_A < \infty$ (as a consequence of the recurrence), however, it can be true that $\mathbb{E}(\tau_A) = \infty$. (e.g. using point 5. of this exercise, one can see that $\mathbb{E}^1(\tau_0) = \infty$ for the random walk on $\mathbb{Z}$ starting at 0). Whereas for $d \geq 3$, we can have $\tau_A = \infty$ with positive probability and still $G_A(x,y) < \infty$ (because there will be a finite number of visits at a given point y).

Solution. The fact that $\tau_A < \infty$ is, for example, a simple consequence of the recurrence of the random walk (the walk will in particular cross any point on the boundary with probability 1). Let us remark also that $G_A(x,y) < \mathbb{E}(\tau_A)$ thus $G_A(x,y) < \infty$ is a consequence of $\mathbb{E}(\tau_A) < \infty$.

We need to prove that $\mathbb{E}(\tau_A) < \infty$. Since for a r.v. X taking value in $\mathbb{N}$ we have

$$\mathbb{E}(X) = \sum_{n\geq 0} n\mathbb{P}(X = n) = \sum_n \sum_{k\leq n}\mathbb{P}(X = n) = \sum_k \sum_{n\geq k}\mathbb{P}(X = n) = \sum_k \mathbb{P}(X \geq k),$$

it is enough to prove that $\mathbb{P}(\tau_A \geq n)$ has a very fast decay to 0 when n goes to infinity.

Yet, using the fact that A is finite, there exists N big enough and $c > 0$ such that

$$\sup_{x\in A}\mathbb{P}^x(\tau_A \geq n) \leq 1 - c \quad \forall n \geq N.$$

Indeed, it is enough to show that there exists N big enough and $c > 0$ such that $\inf_{x\in A}\mathbb{P}^x(\tau_A \leq n) \geq c \quad \forall n \geq N$: for each $x \in A$ we pick a path ω_x which ends in ∂A, then we consider $N = \sup_{x\in A}(|\omega_x|)$ (where $|\omega_x|$ is the length of ω_x) and $c = \inf_{x\in A}\mathbb{P}^x\left((S_n)_{n\leq\tau_A} = \omega_x\right)$.

Then by recurrence we get that

$$\sup_x \mathbb{P}^x\left(\tau_A \geq kn\right) \leq (1-c)^k .$$

Let us just show that it holds for $k=2$.

$$\begin{aligned}
\mathbb{P}^y\left(\tau_A \geq 2n\right) &\leq \sum_z \mathbb{P}^y\left(\tau_A \geq 2n \middle| \tau_A \geq n \text{ and } S_n = z\right) \mathbb{P}^y\left(S_n = z \middle| \tau_A \geq n\right) \mathbb{P}^y\left(\tau_A \geq n\right) \\
&\leq \sum_z \mathbb{P}^z\left(\tau_A \geq n\right) \mathbb{P}^y\left(S_n = z \middle| \tau_A \geq n\right) \mathbb{P}^y\left(\tau_A \geq n\right) \\
&\leq (1-c)^2 \sum_z \mathbb{P}^y\left(S_n = z \middle| \tau_A \geq n\right) \\
&\leq (1-c)^2
\end{aligned}$$

Thus in particular for a given x,

$$\mathbb{P}^x\left(\tau_A \geq kn\right) \leq (1-c)^k$$

and thus

$$\mathbb{E}\left(\tau_A\right) < \infty.$$

(2) Show that $G_A(x,y) = \sum_{n=0}^{\infty} \mathbb{P}^x \left\{S_n = y \text{ and } n < \tau_A\right\}$.

Solution. We can write

$$G_A(x,y) = \mathbb{E}^x\left[\sum_{k=0}^{\infty} \mathbb{1}_{\{S_k = y\} \cap \{\tau_A > k\}}\right] = \sum_{n=0}^{\infty} \mathbb{P}^x\left\{S_n = y \text{ and } n < \tau_A\right\}.$$

(3) Let $y \in A$, and let $f(x) = G_A(x,y)$. Prove that f is a solution of

$$\begin{cases} \Delta f(x) = \begin{cases} -1 & x = y, \\ 0 & x \in A \setminus \{y\} \end{cases} \\ f(x) = 0 & \text{on } \partial A \end{cases}$$

Solution. Checking the boundary conditions is trivial. Let us show the fact that $\Delta f(x) = -\delta_{x=y}$. We have

$$\begin{aligned}
f(x) = G_A(x,y) &= \sum_{n=0}^{\infty} \mathbb{P}^x\left\{S_n = y \text{ and } n < \tau_A\right\} = \sum_{n=0}^{\infty} \sum_{z \sim x} \mathbb{P}^x\left\{S_n = y \text{ and } n < \tau_A \middle| S_1 = z\right\} \mathbb{P}_x\left(S_1 = z\right) \\
&= \delta_{x=y} + \sum_{z \sim x} \mathbb{P}_x\left(S_1 = z\right) \sum_{n=1}^{\infty} \mathbb{P}^x\left\{S_n = y \text{ and } n < \tau_A \middle| S_1 = z\right\} \\
&= \delta_{x=y} + \frac{1}{2d} \sum_{z \sim x} \left(\sum_{n=0}^{\infty} \mathbb{P}^z\left\{S_n = y \text{ and } n < \tau_A\right\}\right) = \delta_{x=y} + \frac{1}{2d} \sum_{z \sim x} G_A(z,y) \\
&= \delta_{x=y} + \frac{1}{2d} \sum_{z \sim x} f(z)
\end{aligned}$$

hence $\Delta f(x) = -\delta_{x=y}$.

(4) Prove that f is the unique solution to the above problem.

Solution. We can apply the point 2. of Exercise 1.

(5) Find an explicit formula for G_A when $d=1$ and $A = \{1, \cdots, n-1\}$.

Solution. We know that $G_A(\cdot, y)$ solves the discrete partial differential equation of point 3. : it is enough to find an explicit formula for the PDE problem. Let us remark that $\Delta f(x) = 0$ implies that f is locally linear. Thus, in order to find $G_A(\cdot, y)$ we need to find a piecewise linear function which is equal to 0 at 0 and n and such that

$$f(y) = \frac{f(y-1) + f(y+1)}{2} + 1. \tag{0.1}$$

Let $a = f(y)$, then the slope at the left of y is $\frac{a}{y}$ and the absolute value of the slope at the right of y is $\frac{a}{n-y}$. By Equation (0.1), a must satisfy :

$$1=\frac{1}{2}\left(\frac{a}{y}+\frac{a}{n-y}\right)$$

or $2y\,(n-y)=a\,(n-y)+ay=an$ thus $a=2\frac{y(n-y)}{n}$. And $G_A\,(.,y)$ is the piecewise linear function such that $G_A\,(0,y)=G_A\,(n,y)=0$ and $G_A\,(y,y)=2\frac{y(n-y)}{n}$.

(6) Prove that the simple random walk on $\mathbb{Z}$ is recurrent.

Solution. Using the point 5., we know that $G_{\{-n+1,\ldots,n-1\}}\,(0,0)=O\,(n)$, but if N_0 is the number of visits at 0 of the simple random walk on $\mathbb{Z}$ and $\mathbb{E}^0$ is the expectation under which the simple random walk on $\mathbb{Z}$ starts from 0, then $\mathbb{E}^0[N_0]=G_{\mathbb{Z}}\,(0,0)=\lim_{n\to\infty}G_{\{-n+1,\ldots,n-1\}}\,(0,0)=\infty$. Thus, the simple random walk on $\mathbb{Z}$ is recurrent.

(7) Show that G_A is the inverse of minus the Laplacian operator $-\Delta$ in the following sense: if $f:A\to\mathbb{R}$ is any function and $h\,(x)=\sum_{y\in A}G_A\,(x,y)\,f\,(y)$, then $-\Delta h=f$

Solution. Let us consider $f:A\to\mathbb{R}$ and $h\,(x)=\sum_{y\in A}G_A\,(x,y)\,f\,(y)$. Then by linearity of the Laplacian:

$$\Delta h\,(x)=\sum_{y\in A}\Delta_x G_A\,(x,y)\,f\,(y)=-\sum_{y}\delta_{x=y}f\,(y)=-f\,(x)$$

hence $-\Delta h=f$.

Exercise 3. *Large deviation estimate and recurrence for the simple random walk on the square lattice* $\mathbb{Z}^2$*.*
Consider the simple random walk $S_k=(X_k,Y_k)$ on $\mathbb{Z}^2$ starting at $(0,0)$, where X_k and Y_k are the coordinates of S_k.

(1) Let $N_{2n}^{(x)}$ be the number of steps in direction x taken by $(S_k)_{k\le 2n}$. Let $\left(S_n^{(1)}\right)_{n\ge 0}$ and $\left(\overline{S}_n^{(1)}\right)_{n\ge 0}$ be two independent one dimensional SRW on $\mathbb{Z}$. Show that

$$\mathbb{P}\left(S_{2n}=0\middle|\left\{N_{2n}^{(x)}=2k\right\}\right)=\mathbb{P}\left(\left\{S_{2k}^{(1)}=0\right\}\text{ and }\left\{\overline{S}_{2n-2k}^{(1)}=0\right\}\right),$$

for all $k,n\in\mathbb{N}$ such that $k\le n$.

Solution. We have (similarly as in Exercise 2.2 from exercise sheet 1)

$$\begin{aligned}\mathbb{P}\left(S_{2n}=0|\left\{N_{2n}^{(x)}=2k\right\}\right) &= \mathbb{P}\left(\{X_{2n}=0\}\text{ and }\{Y_{2n}=0\}\,|\left\{N_{2n}^{(x)}=2k\right\}\right)\\ &= \frac{1}{2^{2n}}\binom{2k}{k}\binom{2n-2k}{n-k}=\mathbb{P}\left(S_{2k}^{(1)}=0\right)\mathbb{P}\left(\overline{S}_{2n-2k}^{(1)}=0\right),\end{aligned}$$

which gives the result using the independence of $\left(S_n^{(1)}\right)_{n\ge 0}$ and $\left(\overline{S}_n^{(1)}\right)_{n\ge 0}$.

(2) Let us suppose that after $2n$ steps for n large enough, the number of steps the walk $(S_k)_{k\ge 0}$ moved in the direction x is the even number in $\{n,n+1\}$. Show that there is a constant $c>0$ such that

$$\mathbb{P}_0\,(S_{2n}=0)>\frac{c}{n}.$$

Solution. We will suppose that n is even. If the assumption is true then, using the previous point, we get

$$\mathbb{P}_0\,(S_{2n}=0)=\mathbb{P}\left(S_{2n}=0|\left\{N_{2n}^{(x)}=n\right\}\right)\sim\left(\frac{1}{2^n}\binom{n}{n/2}\right)^2\sim\frac{c}{n}$$

(3) [*Large deviation estimate*] Show that if N is a $\mathcal{B}in\,(n,p)$ random variable and $q<p$ then there is a constant $c=c(p;q)>0$ such that

$$\mathbb{P}\,(N\le qn)=O\left(e^{-cn}\right).$$

Hint : Think about an inequality which allows to bounds probability with expectations. And since there should be some exponential at the end of the day, try to think about what you can do before applying the inequality.

Solution. For any $\lambda \geq 0$, using Markov inequality, we get

$$\mathbb{P}(N \leq qn) = \mathbb{P}(-\lambda N \geq -\lambda qn) = \mathbb{P}\left(e^{-\lambda N} \geq e^{-\lambda qn}\right) \leq e^{\lambda qn}\mathbb{E}\left[e^{-\lambda N}\right].$$

Recall that the moment generating function of N is $\mathbb{E}\left[e^{-\lambda N}\right] = \left((e^{-\lambda}-1)p+1\right)^n$, thus

$$\mathbb{P}(N \leq qn) \leq e^{n\left(\lambda q + \ln\left((e^{-\lambda}-1)p+1\right)\right)}$$

Let us remark that the function $\phi : \lambda \mapsto \lambda q + \ln\left((e^{-\lambda}-1)p+1\right)$ satisfies $\phi(0) = 0$ and $\phi'(0) = q - p < 0$. Thus, there exists λ such that $\phi(\lambda) < 0$ and thus we found $c = -\phi(\lambda) > 0$ such that

$$\mathbb{P}(N \leq qn) \leq e^{-cn}.$$

(4) Using the intuition built in question (2), the results about 1D SRW and the large deviation estimate to prove that

$$\mathbb{P}_0(S_{2n} = 0) > \frac{c}{n},$$

and thus conclude that S_n is recurrent.

Hint : Use the bound $\mathbb{P}(S_{2n} = 0) \geq \sum_{k=n/2,\,even}^{3n/2} \mathbb{P}\left(S_{2n} = 0 \cap N_{2n}^{(x)} = k\right)$.

Solution. We have

$$\mathbb{P}(S_{2n} = 0) \geq \sum_{k=n/2,\text{even}}^{3n/2} \mathbb{P}\left(S_{2n} = 0 | N_{2n}^{(x)} = k\right)\mathbb{P}\left(N_{2n}^{(x)} = k\right)$$

But if $k = 2k'$ is even in $\left[\frac{n}{2}, \frac{3n}{2}\right]$, then $\mathbb{P}\left(S_{2n} = 0 | N_{2n}^{(x)} = k\right)$ is equal to

$$\frac{1}{4^n}\begin{pmatrix} 2k' \\ k' \end{pmatrix}\begin{pmatrix} 2n-2k' \\ n-k' \end{pmatrix} \sim C\frac{1}{4^n}\frac{(2k')^{2k'+1/2}\,(2(n-k'))^{2(n-k')+1/2}}{k'^{2k'+1}(n-k')^{2(n-k')+1}} \simeq C\frac{1}{\sqrt{k'}\sqrt{n-k'}} \simeq C\frac{1}{n}.$$

Thus

$$\mathbb{P}(S_{2n} = 0) \geq \sum_{k=n/2,\text{even}}^{3n/2} \mathbb{P}\left(S_{2n} = 0 | N_{2n}^{(x)} = k\right)\mathbb{P}\left(N_{2n}^{(x)} = k\right) \simeq C\frac{1}{n}\mathbb{P}\left(N_{2n}^{(x)} \in \left[\frac{n}{2}, \frac{3n}{2}\right] \text{ even}\right).$$

Since according to question (3) (with $p = \frac{1}{2}, q = \frac{1}{4}$) we have $\mathbb{P}\left(N_{2n}^{(x)} \in \left[\frac{n}{2}, \frac{3n}{2}\right] \text{ even}\right) \sim \mathbb{P}\left(N_{2n}^{(x)} \text{ even}\right) = \frac{1}{2}$ (remark that for a symmetric Binomial distribution the probability to be even is equal to the probability to be odd), we obtain that

$$\mathbb{P}(S_{2n} = 0) \geq \frac{c}{n},$$

which allows us to conclude as usual.