Let $A \subseteq \mathbb{Z}^d$ be a finite domain. Recall that the Green's function on A is the function which for each $y \in A$ and each $x \in A \cup \partial A$ takes the value:

$$G_A(x,y) = \mathbb{E}\left[\#\left\{0 \le n < \tau_A : S_n^x = y\right\}\right]$$

where $(S_n^x)_n$ is the simple random walk started at x and $\tau_A = \min\{n \ge 0 : S_n^x \in \partial A\}$.

We also recall that the Harmonic measure on A associated to a subset $B \subseteq \partial A$ is the function which for each $x \in A \cup \partial A$ takes the value:

$$H_A(x,B) = \mathbb{P}\left[S_{\tau_A}^x \in B\right].$$

Exercise 1. *General knowledge*

(1) For $y \in A$, recall what is the discrete PDE satisfied by the function $f(x) = G_A(x,y)$.

Solution. The function $f(x) = G_A(x,y)$ satisfies the discrete PDE $\Delta f(x) = -\delta_{x=y}$, with boundary conditions $f(x) = 0$ for $x \in \partial A$.

(2) For $y \in \partial A$, recall what is the discrete PDE satisfied by the harmonic measure $f(x) = H_A(x,\{y\})$.

Solution. The function $f(x) = H_A(x,\{y\})$ satisfies the discrete PDE $\Delta f(x) = 0$ with boundary conditions $f(x) = \delta_{x=y}$ for $x \in \partial A$.

(3) In this question, a *salary* is a function $s : A \to \mathbb{R}$ and an *exit bonus* is a function $b : \partial A \to \mathbb{R}$. Given a path $\omega = (\omega_0, \dots, \omega_n)$ in $A \cup \partial A$ such that only $\omega_n \in \partial A$, the *reward* associated with ω is $r_{s,b} = \sum_{k=0}^{n-1} s(\omega_k) + b(s_n)$. Give an interpretation of $G_A(x,y)$ and $H_A(x,\{y\})$ as an *expected reward.*

Solution. $G_A(x,y)$ is the expected reward of $(S_1^x, \dots, S_{\tau_A-1}^x)$, with *salary* $\delta_{\cdot=y}$ and *exit bonus* 0:

$$G_A(x,y) = \mathbb{E}^x\left[\sum_{k=0}^{\tau_A-1} \delta_{S_k=y}\right].$$

$H_A(x,\{y\})$ is the expected reward of $(S_1^x, \dots, S_{\tau_A}^x)$, with *salary* 0 and *exit bonus* $\delta_{\cdot=y}$:

$$H_A(x,y) = \mathbb{E}^x\left[\delta_{S_{\tau_A}=y}\right].$$

(4) Give an explicit solution to

$$\begin{cases} \Delta f = 0 & \text{in } A \\ f = F & \text{in } \partial A \end{cases} \tag{0.1}$$

in terms of $\{H_A(x,\{y\})\}_{x,y}$, and give a interpretation of the solution as an *expected reward.*

Solution. The solution is given by

$$f(x) = \sum_{y \in \partial A} H_A(x,\{y\}) F(y).$$

Indeed,

(a) if $x \in \partial A$, $f(x) = \sum_{y \in \partial A} H_A(x,\{y\}) F(y) = \sum_{y \in \partial A} \delta_{x=y} F(y) = F(x)$,
(b) if $x \in A$, $\Delta f(x) = \sum_{y \in \partial A} (\Delta H_A(x,\{y\})) F(y) = \sum_{y \in \mathrm{A}} 0 F(y) = 0$,
(c) we have a unique solution since if f_1 and f_2 are solutions, then $h = f_1 - f_2$ is harmonic and null on ∂A: by the *maximum principle,* $h = 0$ and thus $f_1 = f_2$.

The solution f at x can be seen as the expected reward of $(S_1^x, \dots, S_{\tau_A}^x)$, starting at x, with *salary* 0 and *exit bonus* F:

$$f(x) = \mathbb{E}\left[F\left(S_{\tau_A}^x\right)\right].$$

(5) Solve

$$\begin{cases} \Delta f = \rho & \text{in } A \\ f = 0 & \text{in } \partial A \end{cases} \tag{0.2}$$

in terms of the Green's function and give an interpretation of $f(x)$ as an *expected reward.*

Solution. The unique solution is given by

$$f(x) = -\sum_{y\in A} \rho(y)\, G_A(x,y).$$

Indeed,

(a) if $x \in A$, $\Delta f(x) = -\sum_{y\in A} \rho(y)\,(\Delta G_A(\cdot,y))(x) = \sum_{y\in A} \rho(y)\,\delta_{x=y} = \rho(x)$.
(b) if $x \in \partial A$, $f(x) = \sum_{y\in A} \rho(y)\, G_A(x,y) = \sum_{y\in A} \rho(y)\, 0 = 0$.

The solution f at x can be seen as the expected reward of $(S_{n\wedge\tau_A})_n$, starting at x, with *salary* $-\rho$ and *exit bonus* 0:

$$f(x) = -\mathbb{E}\left[\sum_{k=0}^{\tau_A-1} \rho(S_k^x)\right].$$

(6) Explain why

$$G_A(x,x) = \sum_{\omega: x\to x, \omega\subset A} \left(\frac{1}{2d}\right)^{|\omega|}$$

where $|\omega|$ is the length of the path $\omega = \left[x=\omega_0, \cdots, \omega_{|\omega|} = x\right]$.

Solution. We have

$$\begin{aligned} G_A(x,x) = \mathbb{E}\left[\#\{0\le n<\tau_A :\ S_n^x = x\}\right] = \mathbb{E}\left[\sum_{n<\tau_A} \mathbb{1}_{S_n^x=x}\right] &= \mathbb{E}\left[\sum_n \mathbb{1}_{\{S_n^x=x, n<\tau_A\}}\right] \\ &= \sum_n \mathbb{P}(S_n^x = x, n<\tau_A) \\ &= \sum_n \sum_{\omega: x\to x, \omega\subseteq A, |\omega|=n} \mathbb{P}(\omega) \\ &= \sum_{\omega: x\to x, \omega\subseteq A} \left(\frac{1}{2d}\right)^{|\omega|} \end{aligned}$$

Exercise 2. *Discretisation of PDEs : the equilibrium case*
We want to study the discrete PDEs :

$$\begin{cases} \Delta f = \rho & \text{in } A \\ f = F & \text{in } \partial A \end{cases} \tag{0.3}$$

and to give an explicit formulation in terms of the given functions ρ, F, the Green's function G_A and the harmonic measure $H_A(x,y)$

(1) Recall why there is at most one solution to the system (0.3).

Solution. If f_1 and f_2 are two solutions of the discrete PDEs (0.3), then $h = f_1 - f_2$ is harmonic and null on ∂A: by the *maximum principle,* $h = 0$ and thus $f_1 = f_2$.

(2) Solve the system (0.3) and give an interpretation of $f(x)$ as an *expected reward.*

Solution. If we consider a solution f_1 of the discrete PDE (0.1), and f_2 a solution of the discrete PDE (0.2), then $f_1 + f_2$ is a solution of the discrete PDE (0.3) (and actually the unique one by the point 1.)
Thus the unique solution of (0.3) is given by

$$f(x) = -\sum_{y\in A} \rho(y)\, G_A(x,y) + \sum_{y\in\partial A} H_A(x,\{y\})\, F(y).$$

The solution f at x can be seen as the expected reward of $(S_{n\wedge\tau_A})_n$, starting at x, with *salary* $-\rho$ and *exit bonus* F:

$$f(x) = \mathbb{E}^x\left[-\left[\sum_{k=0}^{\tau_A-1} \rho(S_k)\right] + F(S_{\tau_A})\right].$$

Exercise 3. *Discretisation of PDEs: the evolution case*

We want to give an explicit formulation and a probabilistic interpretation of the solution to the discrete partial differential equation:

$$\begin{cases} f(x,t+1) - f(x,t) = \Delta f(x,t) & \text{for } (x,t) \in A \times \mathbb{N} \\ f(x,t) = F(x) & \text{for } (x,t) \in \partial A \times \mathbb{N} \cup A \times \{0\} \end{cases} \tag{0.4}$$

where $f : (A \cup \partial A) \times \mathbb{N} \to \mathbb{R}$.

(1) Prove that the solution to (0.4) is unique.

Solution. At time $t = 0$, f is uniquely defined by F. For consecutive time-steps, we have $f(x,t+1) = f(x,t) + \Delta f(x,t)$.

(2) Suppose that $f(\cdot,t)$ converges to a function $g(\cdot)$ when t goes to infinity. What discrete partial differential equation does g satisfy? Thus, which function (or modification of it) should appear in the explicit formulation: the Harmonic measure or the Green's function?

Solution. If $f(\cdot,t)$ converges to a function $g(\cdot)$ then taking $t \to \infty$ in the discrete PDE, we get : $\Delta g(x)$=0. Thus, we should consider a modification of the Harmonic measure.

(3) Write the discrete PDE as $\Delta_t f(x,t) = 0$ where Δ_t is a linear operator.

Solution. The Laplacian is $\Delta f(x,t) = \left(\frac{1}{2d}\sum_{y\sim x} f(y,t)\right) - f(x,t)$. Thus we can write : $\Delta_t f(x,t) = \left(\frac{1}{2d}\sum_{y\sim x} f(y,t)\right) - f(x,t+1) = 0$.

(4) Find an explicit formulation of a solution. *Hint: For $t \in \mathbb{N}$ consider the random variable $S_{\tau_A \wedge t}$* where $\tau_A \wedge t = min\{\tau_{A,}t\}$ and take its expected value under the image of F.

Solution. The last question implies that

$$f(x,t+1) = \frac{1}{2d}\sum_{y\sim x} f(y,t)$$

In particular, for $t = 1$ and $x \in A$ we have: $f(x,1) = \frac{1}{2d}\sum_{y\sim x} F(y)$. Similarly, for $t = 2$ and $x \in A$ we have:

$f(x,2) = \frac{1}{2d}\sum_{x'\sim x, x'\in A} f(x',1) + \frac{1}{2d}\sum_{x'\sim x, x'\in \partial A} f(x',1) = \frac{1}{(2d)^2}\sum_{x'\sim x, x'\in A}\sum_{y\sim x'} f(y,0) + \frac{1}{2d}\sum_{y\sim x, y\in\partial A} F(y)$
$= \sum_{y\in A}\mathbb{P}(S_2^x = y, \tau_A > 2)F(y) + \sum_{y\in\partial A}\mathbb{P}(S_{\tau_A}^x = y, \tau_A \le 2)F(y) = \mathbb{E}\left(F\left(S_{\tau_A\wedge 2}^x\right)\right)$

It is then natural to consider

$$f(x,t) = \mathbb{E}\left(F\left(S_{\tau_A\wedge t}^x\right)\right)$$

where the random walk gets a reward if either it exits the set A or it runs out of time.

Then $f(x,t) = \mathbb{E}\left(F\left(S_{\tau_A\wedge t}^x\right)\right) = \frac{1}{2d}\sum_{y\sim x}\mathbb{E}\left(F\left(S_{\tau_A\wedge t}^x\right)|S_1^x = y\right)$, and

$$\mathbb{E}\left(F\left(S_{\tau_A\wedge t}^x\right)|S_1^x = y\right) = \mathbb{E}\left(F\left(S_{\tau_A\wedge(t-1)}^y\right)\right)$$

by the Markov property thus

$$f(x,t) = \frac{1}{2d}\sum_{y\sim x} f(y,t-1).$$

(5) Let us consider the oriented graph $A^{\rightarrow} = A \times \mathbb{N} \subseteq \mathbb{Z}^{d+1}$ with neighbours of the form $(x_1,t_1) \rightsquigarrow (x_2,t_2)$ if and only if $x_1 \sim x_2$ in A and $t_2 = t_1 + 1$ ($\rightsquigarrow$ represents the oriented edge pointing from (x_1,t_1) to (x_2,t_2)). We define the Laplacian on $A^{\rightarrow}$ for a function $f : A \cup \partial A^{\rightarrow} \to \mathbb{R}$ as

$$\Delta f(\bar{x}) = \frac{1}{\#\{\bar{y} \rightsquigarrow \bar{x}\}}\sum_{\bar{y}\rightsquigarrow\bar{x}}\left(f(\bar{y}) - f(\bar{x})\right).$$

(a) What is $\partial A^{\rightarrow}$?

Solution. $\partial A^{\rightarrow} = (A \times \{0\}) \sqcup (\partial A \times \mathbb{N})$.

(b) Show that f is a solution to (0.4) if and only if f is harmonic on $A^{\rightarrow}$ with suitable boundary conditions.

Solution. We have seen that the function f is a solution of (0.4) if and only if $f(x,t+1) = \frac{1}{2d}\sum_{y \rightsquigarrow x} f(y,t)$ and it satisfies the same boundary conditions. Let us remark that this last equation can be written as

$$f(x,t+1) = \frac{1}{2d} \sum_{(y,t)\rightsquigarrow(x,t+1)} f(y,t)$$

which is exactly equivalent to the fact that f is harmonic on $A^{\rightarrow}$.

(c) Show that the harmonic measure $H_{A^{\rightarrow}}((x,t),\{(y,s)\})$ is equal to

$$\begin{cases} \mathbb{P}^x\left(S_{\tau_A} = y \text{ and } \tau_A = t-s\right) & \text{if } s>0 \\ \mathbb{P}^x\left(S_t = y \text{ and } \tau_A \geq t\right) & \text{if } s=0 \end{cases}$$

where we recall that $(S_n^x)_{n=0}^{\infty}$ is the simple random walk on A starting at x.

Solution. The harmonic measure $H_{A^{\rightarrow}}((x,t),\{(y,s)\})$ is equal to $\mathbb{P}\left(S^{\rightarrow(x,t)}_{\tau_{A^{\rightarrow}}} = (y,s)\right)$, where $S^{\rightarrow}_{\tau_{A^{\rightarrow}}}$ is the *inverse* simple random walk on $A^{\rightarrow}$ (i.e. it can only jump in the inverse direction of any oriented edge). Let us consider the random walk $\left(S^{\rightarrow(x,t)}_{n\wedge\tau_A^{\rightarrow}}\right)_{n\in\mathbb{N}}$ starting from (x,t) and stopped when it hits $\partial A^{\rightarrow}$. If the simple random walk stops when it hits $\partial A^{\rightarrow}$ and goes out at (y,s) with $s>0$ it means that $\tau_A^{\rightarrow} = \tau_A$ and $\tau_A < t$, thus

$$\mathbb{P}^{(x,t)}\left(S^{\rightarrow}_{\tau_{A^{\rightarrow}}} = (y,s)\right) \quad = \quad \mathbb{P}^x\left(S_{\tau_A} = y \text{ and } \tau_A = t-s\right)$$

and if the simple random walk stops when it hits $\partial A^{\rightarrow}$ and goes out at (y,s) with $s=0$ it means that $\tau_{A^{\rightarrow}} = t$ and actually $\tau_A \geq t$ thus

$$\mathbb{P}^{(x,t)}\left(S^{\rightarrow}_{\tau_{A^{\rightarrow}}} = (y,s)\right) \quad = \quad \mathbb{P}^x\left(S_t = y \text{ and } \tau_A \geq t\right).$$

(d) Using the last question, give the explicit formulation of (0.4).

Solution. We know that the unique harmonic function f on $A^{\rightarrow}$ with boundary conditions given by $f(x,t) = F(x)$ for $(x,t)\in\partial A^{\rightarrow}$ is given by

$$f((x,t)) = \sum_{(y,s)\in\partial A^{\rightarrow}} H_{A^{\rightarrow}}((x,t),(y,s))\,F(y).$$

Using the last question, we can write it as

$$f((x,t)) = \sum_{y\in A} \mathbb{P}^x\left(S_t = y \text{ and } \tau_A \geq t\right)F(y) + \sum_{y\in\partial A}\sum_{s=1}^{t}\mathbb{P}^x\left(S_{\tau_A} = y \text{ and } \tau_A = t-s\right)F(y).$$

Let us remark that $\sum_{s=1}^{t}\mathbb{P}^x\left(S_{\tau_A} = y \text{ and } \tau_A = t-s\right) = \mathbb{P}\left(S_{\tau_A} = y \text{ and } \tau_A < t\right)$ thus :

$$\begin{aligned} f((x,t)) &= \sum_{y\in A}\mathbb{P}^x\left(S_t = y \text{ and } \tau_A \geq t\right)F(y) + \sum_{y\in\partial A}\mathbb{P}^x\left(S_{\tau_A} = y \text{ and } \tau_A < t\right)F(y) \\ &= \sum_{y\in A\sqcup\partial A}\mathbb{P}^x\left(S_{t\wedge\tau_A} = y\right)F(y) \\ &= \mathbb{E}^x\left(F\left(S_{t\wedge\tau_A}\right)\right) \end{aligned}$$

and thus we recover the result of point 3.

Exercise 4. *Discretisation of PDEs: the time-dependent boundary condition.*

We want to give an explicit formulation and a probabilistic interpretation of the solution to the discrete partial differential equation:

$$\begin{cases} \Delta f(x,t) = f(x,t+1) - f(x,t) & \text{for } (x,t)\in A\times\mathbb{N} \\ f(x,t) = F(x,t) & \text{for } (x,t)\in\partial A\times\mathbb{N}\cup A\times\{0\} \end{cases}$$

where $f : A\cup\partial A \to \mathbb{R}$.

Following the same ideas used for the point 4. of Exercise 3, give an explicit formulation and a probabilistic interpretation of the solution to the latter discrete partial differential equation.

Solution. For $z \in \mathbb{Z}$ we define $(z)^+ := max\{z, 0\}$.

Using the same ideas used for the point 4. of Exercise 3, we get that the solution of this discrete PDE is given by:

$$f(x,t) = \mathbb{E}\left(F\left(S_{\tau_A \wedge t}, (t-\tau_A)^+\right)\right).$$

Indeed, we are still looking for an harmonic function on $A^{\rightarrow}$ but now the boundary conditions are different : if the walk starts at (x,t) and goes out at (y,s), then the reward is $F(y,s)$. But $s = t - \tau_A$ if $\tau_A < t$ and $s = 0$ if $\tau_A \geq 0$: thus $s = (t-\tau_A)^+$.