EXERCISE SHEET 10: GLOBAL SIMULATION OF CELLULAR AUTOMATA

Exercise 1 (Transforming subshifts of finite type I)**.** *Let S be a finite set, $d \in \mathbb{N}^+$, and let $\Sigma_{\mathcal{F}_1}, \Sigma_{\mathcal{F}_2} \subseteq S^{\mathbb{Z}^d}$ be two subshifts of finite type. Show that $\Sigma_{\mathcal{F}_1} \cap \Sigma_{\mathcal{F}_2}$ is also a subshift of finite type and express its set of forbidden words in terms of $\mathcal{F}_1$ and $\mathcal{F}_2$. Extra challenge: Is $\Sigma_{\mathcal{F}_1} \cup \Sigma_{\mathcal{F}_2}$ also a subshift of finite type?*

Exercise 2 (Transforming subshifts of finite type II)**.** *Let $\Phi : S^{\mathbb{Z}^d} \to T^{\mathbb{Z}^d}$ be a cellular transformaton between two configuration spaces. Suppose $\Sigma \subseteq T^{\mathbb{Z}^d}$ be a subshift of finite type. Is $\Phi^{-1}(\Sigma)$ also a subshift of finite type? Prove or find a counterexample.*

Exercise 3 (Surjective Simulation)**.** *Attempt to prove that a surjective CA can only globally simulate surjective cellular automata. At which point does the argument fail? What if we required the decoder to have a full domain?*

Exercise 4 (Weaker version of global simulation)**.** *Consider the elementary CAs Rule 4 and Rule 76, with global functions F and G respectively.*

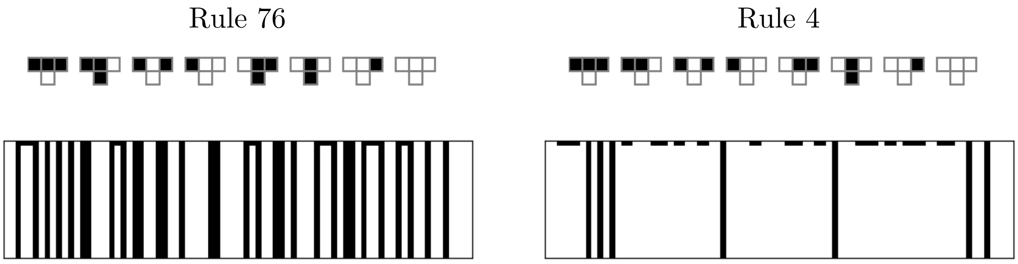

Figure 1: Rule tables and space-time diagrams of elementary CAs 76 and 4.

The dynamics of both CAs is relatively simple, since they are both idempotent; $F^2 = F$ and $G^2 = G$. Whereas Rule 4 only preserves blocks of the form 010, Rule 76 acts as identity for all neighbourhoods except for 111. We will discuss that Rule 76 can simulate Rule 4. We define two local mappings:

$$e : \mathbf{2} \to \mathbf{2}^2 \qquad 0 \mapsto 00 \qquad 1 \mapsto 11.$$

$$d : \mathbf{2}^2 \to \mathbf{2} \qquad 00, 01, 10 \mapsto 0 \qquad 11 \mapsto 1.$$

We define the encoding $\mathcal{E} : \mathbf{2}^{\mathbb{Z}} \to \mathbf{2}^{\mathbb{Z}}$ that maps each configuration, cell by cell, using the local map e and concatenating the cell tuples. Similarly, we define the decoder $\mathcal{D} : \mathbf{2}^{\mathbb{Z}} \to \mathbf{2}^{\mathbb{Z}}$ that first "partitions" each configuration into blocks of two consecutive cells $c_{2n}c_{2n+1}$, and uses the local map d to decode each such supercell. Formally, $\mathcal{D} = \tilde{D} \circ o_{\langle D,V \rangle}$ where $D = \{0, 1\}$ and $V = (2)$ determine the tiling of $\mathbb{Z}$ and $\tilde{\mathcal{D}} : (\mathbf{2}^2)^{\mathbb{Z}} \to \mathbf{2}^{\mathbb{Z}}$ is a cellular transformaton on the packed space $(\mathbf{2}^2)^{\mathbb{Z}}$ with radius 0 and local rule d. The situation is illustrated in Figure 2.

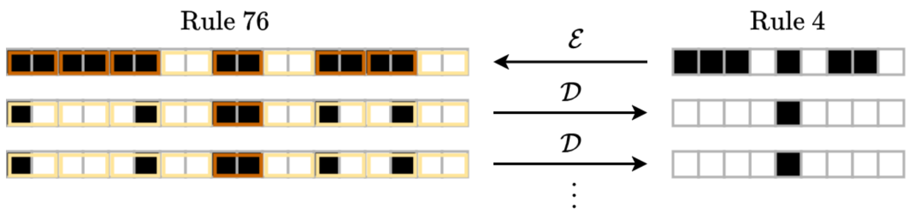

Figure 2: Rule 76 weakly globally simulating Rule 4.

(i) *Show that for each initial configuration $c \in \mathbf{2}^{\mathbb{Z}}$ and for any number of iterations of n, it holds that $\mathcal{D} \circ G^n \circ \mathcal{E}(c) = F^n(c)$.*

(ii) *Show that it does not hold that $\mathcal{D} \circ G \circ \mathcal{D}^{-1} = F$, and therefore, that Rule 76 does not globally simulate Rule 4 with this exact encoder and decoder.*

(iii) *Can you modify the decoder in such a way, that it does witness Rule 76 globally simulating Rule 4? Hint: When does the decoding fail? Can you forbid some finite patterns to restrict the domain of $\mathcal{D}$?*

Exercise 5 (Understanding Tilings and Packings)**.** *Let $D = \{(0,0),(0,1),(1,1),(1,2)\} \subseteq \mathbb{Z}^2$ and $V = ((1,-1),(1,3)) \in (\mathbb{Z}^2)^2$. Let $E = \{(0,0),(1,0)\} \subseteq \mathbb{Z}^2$ and $W = ((2,0),(1,1))$.*

(i) *Draw (D,V) and (E,W) to check they are both valid tilings of $\mathbb{Z}^2$.*

(ii) *What is the tiling corresponding to the composition of the packing operators $o_{\langle E,W\rangle} \circ o_{\langle D,V\rangle}$? Draw the tile and determine its corresponding basis.*

(iii) *What is the tiling corresponding to the composition of the packing operators $o_{\langle D,V\rangle} \circ o_{\langle E,W\rangle}$? Draw the tile and determine its corresponding basis.*

Exercise 6 (Reflection as a packing)**.** *Suppose $\mathcal{A}$ is an elementary cellular automaton, and suppose $\mathcal{B}$ is the elementary cellular automaton obtained from $\mathcal{A}$ by exchanging the roles of the left and right neighbours. We discussed that $\mathcal{A}$ and $\mathcal{B}$ are conjugate via the map $\Pi : \mathbf{2}^{\mathbb{Z}} \to \mathbf{2}^{\mathbb{Z}}$ defined by $\Pi(c)(i) = c(-i)$. Find a clever tiling which shows that $\mathcal{A}$ globally simulates $\mathcal{B}$, and vice versa.*

Exercise 7. *Prove the "packed version of Curtis-Hedlund-Lyndon Theorem": Let S, T be finite sets and $V = (\mathbf{v}_1, \ldots, \mathbf{v}_d)$, $W = (\mathbf{w}_1, \ldots, \mathbf{w}_d)$ two sequences of d linearly independent vectors in $\mathbb{Z}^d$; we denote their corresponding matrices by M_V and M_W. Let (D,V) and (E,W) be two tilings of $\mathbb{Z}^d$ and let $\Phi : S^{\mathbb{Z}^d} \to T^{\mathbb{Z}^d}$ be an arbitrary map. Then, the following two conditions are equivalent:*

(i) *Φ is continuous and for each $\mathbf{v} \in \mathbb{Z}^d$ satisfies $\Phi \circ \sigma_{M_V \mathbf{v}} = \sigma_{M_W \mathbf{v}} \circ \Phi$.*

(ii) *$\Phi^{\langle D,V\rangle}_{\langle E,W\rangle} := o_{\langle E,W\rangle} \circ \Phi \circ \left(o_{\langle D,V\rangle}\right)^{-1} : (S^D)^{\mathbb{Z}^d} \to (T^E)^{\mathbb{Z}^d}$ is a cellular transformaton.*