EXERCISE SHEET 2: UNIVERSAL TURING MACHINES AND UNDECIDABILITY

Exercise 1 (Encoding tuples of sequences). *Define a bijection $\gamma : \Sigma^* \times \Sigma^* \to \mathbb{N}$ such that both γ and its inverse are total computable functions.*

Exercise 2 (Applications of the Parameter Theorem). *Let $d : \mathbb{N} \to \mathbb{N}$ be a function defined as follows:*

$$\varphi_{d(u)}(z) = \begin{cases} \varphi_{\varphi_u(u)}(z) & \text{if } \varphi_u(u) \text{ is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Using the Parameter Theorem, verify that the function d is indeed well-defined, and that it is an injective total computable function.

Exercise 3 (Different points of view on recursively enumerable sets). *Let $S \subseteq \mathbb{N}$. Recall the definition of a recursively enumerable set: S is called* recursively enumerable *if there exists a partial computable function f such that for all $s \in \mathcal{S}$ $f(s) = 1$ and $f(s)$ need not be defined or is different from 1 for $s \notin S$. Prove that each of the following definitions is equivalent to it.*

1. *There exists a Turing machine that, from an empty input, keeps progressively listing exactly all elements of S on the tape.*
2. *There exists a partial computable function f such that $S = \operatorname{dom}(f)$.*
3. *There exists a partial computable function f such that $S = \operatorname{rng}(f)$.*

Exercise 4 ("Harder" than the halting problem). *We have shown that the halting problem $H = \{n \mid \varphi_n(n) \text{ is defined}\}$ is not decidable. Show that it is recursively enumerable. Let $S \subseteq \mathbb{N}$ and $S^{\mathrm{C}} \subseteq \mathbb{N}$ denote its complement. Show that if both S and S^{C} are recursively enumerable, then S is decidable. Use this to show that the complement of the halting problem H^{C} is not recursively enumerable.*

Exercise 5 (Undecidability via diagonalization). *Using the diagonalization argument, show that the set $S = \{n \in \mathbb{N} \mid \varphi_n : \mathbb{N} \to \mathbb{N} \text{ is a total computable function}\}$ is not decidable.*

Exercise 6 (Undecidability via reduction). *Using the reduction technique, show that the set $\{n \in \mathbb{N} \mid \operatorname{dom}(\varphi_n) = \emptyset\}$ is not decidable.*