Exercise sheet 6: More Examples of CA Self-Replicators

Exercise 1 (Interplay between CA neighbourhood and state space size)**.** *Consider a 2D CA $\mathcal{A} = (S^{\mathbb{Z}^2}, F)$ with a neighbourhood formed by three nearest neighbours of a cell, in all 8 directions, forming a 7×7 neighbourhood, as shown in Figure 1 (a). Define a geometrical transformation of the grid, which yields a new 2D CA with a larger state space but with the Moore neighbourhood. Verify that the dynamics of the two CAs is equivalent (up to the geometrical transformation). Hint: Figure 1 (b).*

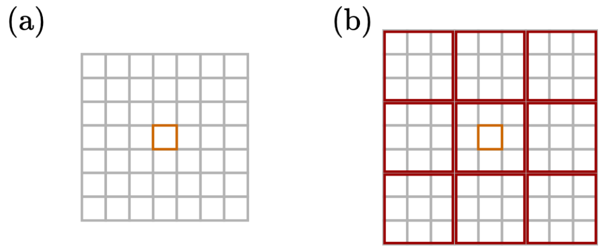


Figure 1: Illustration of a packing of the CA space. (a) Original CA with state space S, and a 7×7 neighbourhood. (b) Packed CA with state space S^9 and a 3×3 neighbourhood.

Counting objects up to symmetries

It will often be useful for us to count the number of unique object up to some group of symmetries. For that, it will be very useful to apply Burnside's lemma that we recall below.

Lemma (Burnside's Lemma)**.** *Let G be a finite group acting on a set X. For $g \in G$, let*

$$X^g = \{x \in X \mid g(x) = x\}$$

denote the set of elements of X that are fixed by g and for $x \in X$ let

$$\mathcal{O}_x = \{y \in X \mid \text{ there exists } g \in G \text{ such that } g(x) = y\}$$

denote the orbit of $x \in X$. Then the number of unique orbits, denoted $|X/G|$, is given by

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

The next two exercises can be solved by a straightforward application of the lemma.

Exercise 2 (Cube colourings)**.** *Consider a cube whose side's we aim to colour using either yellow or blue. How many unique colourings are there up to rotations of the cube?*

Exercise 3 (Unique neighbourhoods in Codd's CA)**.** *Consider a two-dimensional cellular automaton with a von Neumann neighbourhood and 8 states.*

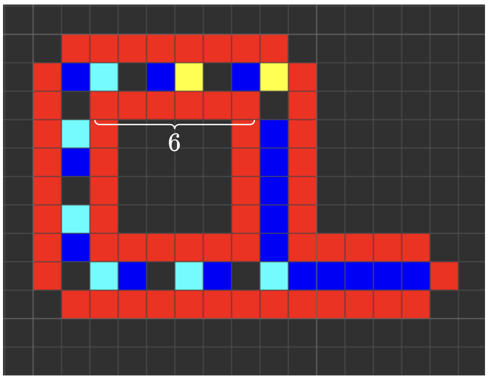
(1) In the absence of any symmetries, how many outputs of the local rule must be specified?

(2) If the CA is invariant under rotations by 90°, 180°, and 270°, how does this number change?

Exercise 4 (Investigating Langton's loop)**.** *For this exercise we encourage you to use Golly, either in the browser at* `https://golly.sourceforge.io/webapp/golly.html` *or download it as an app from* `https://golly.sourceforge.io/`*. In both cases, navigate to the "Loops" folder, and open the "Langtons_loops.rle" file. Exploring this CA interactively should help you answer the following questions.*

(a) Let us measure the size of a square shaped loop by the number of sheath cells in one side of its inner sheath. For instance, the original loop below has size 6. Can you design self-replicating loops in Langton's CA of growing sizes? Which loop sizes are feasible for Langton's design of self-replication and which are not?

(b) Can you design a self-replicating loop in Langton's CA which has a shape different from a square? E.g., can there be a self-replicating rectangle?



Barricelli's Model

Exercise 5 (Dealing with reproduction)**.** *What is the next iteration of this configuration?*

…			2		$\underline{2}$			…

Exercise 6 (A larger self-replicator)**.** *Compute two iterations of the following configuration to verify that it can self-replicate.*

…		5	$\underline{3}$	1	$\underline{3}$		$\underline{3}$	1		…

Exercise 7. *Is the Barricelli's model a cellular automaton?*