Exercise sheet 7: A First Look at the Space of Cellular Automata

Exercise 1 (Checking total periodicity). *Construct a configuration $c \in \mathbf{2}^{\mathbb{Z}^2}$ which is invariant under $\sigma_{\mathbf{v}}$ with $\mathbf{v} = (-1, 1)$ but which is not totally periodic.*

Exercise 2 (Finite cyclic configurations and totally periodic configurations). *Let $\mathcal{A} = (S^{\mathbb{Z}^d}, F)$ be a cellular automaton and $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$. We can "unwrap" each finite cyclic configuration $\tilde{c} \in S^{\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_d}}$ to obtain a totally periodic configuration $c \in S^{\mathbb{Z}^d}$ where for each $(i_1, i_2, \ldots, i_d) \in \mathbb{Z}^d$ we define*

$$c(i_1, i_2, \ldots, i_d) = \tilde{c}(i_1 \bmod m_1, i_2 \bmod m_2, \ldots, i_d \bmod m_d).$$

Let us define the subspace of all periodic configurations obtained in this way by $C_{\mathbf{m}}$. Show that the two dynamical systems are isomorphic:

$$F\big|_{C_{\mathbf{m}}} : C_{\mathbf{m}} \to C_{\mathbf{m}} \quad \textit{and} \quad F_{\boldsymbol{m}} : S^{\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_d}} \to S^{\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_d}}.$$

Is it also the case that every totally periodic configuration can be expressed as an unwrapped version of some finite cyclic configuration?

Exercise 3 (Counting totalistic CAs). *How many outer totalistic CAs are there in two dimensions, with radius $r = 1$ and two states?*

Exercise 4 (Computing the dual). *Let $f : \mathbf{2}^3 \to \mathbf{2}$ be an ECA local rule and let $g = \sigma(f)$ be the dual local rule obtained by exchanging the roles of 0 and 1. Suppose the binary expression of the Wolfram number of f is $(a_7a_6a_5a_4a_3a_2a_1a_0)_2$. What is the binary expression of the Wolfram number of g?*

Exercise 5 (Equivalences classes of ECAs). *Let G be the group acting on* **ECA** *generated by π and σ.*

(a) Compute the elements of G explicitly.

(b) Let $x \in G$ and $f \in$ **ECA**. *Show that the ECAs with local rules f and $x(f)$ are conjugated.*

(c) using the Burnside Lemma, compute the number of unique ECAs up to the equivalence induced by G.

Exercise 6 (Equivalences of 2D CAs). *Consider the class of two-dimensional CAs with von Neumann neighbourhood and two states. How many CAs are in this class? What are the natural symmetries of their local rules to consider? (Here, by "natural" we mean that it easily follows that two CAs in the same equivalence class are isomorphic). Using the hint that there are 16 such symmetries, list all of them. How many unique CAs up to the equivalence does it induce?*

Exercise 7 (Fuzziness of Wolfram classes). *Assume that Rule 110 belongs to Wolfram's Class 4 and Rule 30 belongs to Class 3. Can you combine the two to create a new 1D CA with radius $r = 1$ and a richer state set S for which the Wolfram's Class is unclear?*