EXERCISE SHEET 8: TOPOLOGICAL DYNAMICS OF CELLULAR AUTOMATA I

Exercise 1 (Cylinder Sets)**.** *Let $D \subset \mathbb{Z}^d$ be a finite domain and $c \in S^{\mathbb{Z}^d}$ that together define the cylinder set* $\operatorname{cyl}(c, D) = \{c' \in S^{\mathbb{Z}^d} \mid c(i) = c'(i) \text{ for all } i \in D\}$*. Prove that this set is both open and closed.*

Exercise 2 (Compactness)**.** *Prove that the metric space $(S^{\mathbb{Z}^d}, \mathrm{d})$ is compact by showing that every sequence has a converging subsequence.*

Exercise 3 (Both assumptions in the Curtis–Hedlund–Lyndon Theorem are essential)**.** *Construct an example of a map $F : \mathbf{2}^{\mathbb{Z}} \to \mathbf{2}^{\mathbb{Z}}$ such that:*

(a) F is continuous, but is not a global map of any CA.

(b) F commutes with every shift operator but is not a global map of any CA.

Exercise 4 (Alternative proof of Myhill's direction)**.** *Recall that for a CA $\mathcal{A} = (S^{\mathbb{Z}^d}, F)$ we denote by* $\mathrm{Per}_{\mathrm{tot}}$ *the set of its totally periodic configurations. Consider the relationships between the following notions.*

(a) F is injective, $F\big|_{\mathrm{Per}_{\mathrm{tot}}}$ is injective. (One implication is easy to prove, which one is it? Prove it. Do not worry about the converse.)

(b) F is surjective, $F\big|_{\mathrm{Per}_{\mathrm{tot}}}$ is surjective. (One implication is easy to prove, which one is it? Prove it. Do not worry about the converse.)

*(c) $F\big|_{\mathrm{Per}_{\mathrm{tot}}}$ is injective, $F\big|_{\mathrm{Per}_{\mathrm{tot}}}$ is surjective. (Which implications hold here? Prove them.)**

Use the relationships you proved to give an alternative proof of the implication

$$F \text{ injective} \implies F \text{ surjective.}$$

Exercise 5 (Permutivity implies surjectivity)**.** *Show that every left-permutive 1D CA is surjective.*

Let $\mathcal{A} = (S^{\mathbb{Z}}, F)$ be a 1D CA with radius r and local rule $f : S^{2r+1} \to S$. We now introduce the de Bruijn diagram as an alternative representation of the local rule f, which will become very useful when deciding whether $\mathcal{A}$ is injective or surjective. The *de Bruijn diagram* of $\mathcal{A}$ is the directed edge-labelled graph defined as follows:

- the set of vertices is S^{2r};
- for each $s_1 s_2, \cdots s_{2r} s_{2r+1} \in S^{2r+1}$ there is an edge from vertex $s_1 s_2 \cdots s_{2r-1} s_{2r}$ to the vertex $s_2 s_3 \cdots, s_{2r} s_{2r+1}$ labelled by $f(s_1, s_2, \ldots, s_{2r+1})$.

Exercise 6 (De Bruijn diagrams I)**.** *Construct the de Bruijn diagram of Rule 30 and Rule 110.*

Exercise 7. *Fix a 1D CA $\mathcal{A} = (S^{\mathbb{Z}}, F)$ and verify the following properties:*

(a) Every bi-infinite path in the de Bruijn diagram determines a configuration $c \in S^{\mathbb{Z}}$, and the sequence of its edge labels is exactly $F(c)$.

**Hint: You can further partition the space* $\mathrm{Per}_{\mathrm{tot}} = \bigcup_{i=1}^{\infty} C_i$ *for some suitable finite sets C_i such that $F\big|_{C_i} : C_i \to C_i$. What are those sets?*

(b) $\mathcal{A}$ is injective if and only if every two different bi-infinite paths have different labels.

(c) $\mathcal{A}$ is surjective if and only if for every $c \in S^{\mathbb{Z}^d}$ there is a path labeled by c.

(d) $\mathcal{A}$ is surjective if and only if there exists no diamond in the de Bruijn diagram; that is, two different finite paths in the graph that start in the same vertex and end in the same vertex, and have the same labels.

What do orphans correspond to in the de Bruijn graph?

Exercise 8 (Orphans of a CA form a regular language)**.** *Transform the de Bruijn diagram of Rule 110 into a larger directed graph which acts as a finite state automaton accepting exactly all orphans of Rule 110.*

Hint: Each vertex of the new graph is a subset of $\mathbf{2}^2$, the initial state being $\{00, 01, 10, 11\}$.