EXERCISE SHEET 9: TOPOLOGICAL DYNAMICS OF CELLULAR AUTOMATA II

Exercise 1 (Finding equicontinuity points)**.** *Consider the ECA Rule 128 whose local rule* $f : \mathbf{2}^3 \to \mathbf{2}$ *can be expressed as* $f(a,b,c) = abc \bmod 2$*. Find an equicontinuity point of this CA. Is the CA equicontinuous?*

Exercise 2 (Flawed argument for why dimensionality does not matter)**.** *For each finite set* S *and dimension* $d \in \mathbb{N}$ *the metric space we defined on* $S^{\mathbb{Z}^d}$ *satisfies the following:*

(i) it is compact

(ii) it is perfect (every point can be approximated arbitrarily well by other points from the set)

(iii) it has dimension zero (it has a basis consisting of sets that are both closed and open)

(iv) it is non-empty.

Due to Brouwer's Theorem, every metric space satisfying the above properties is homeomorphic to the standard Cantor set (the set obtained from the unit interval by iteratively excluding the middle third). As a special case, we get that $S^{\mathbb{Z}^d}$ *and* $S^{\mathbb{Z}}$ *are homeomorphic spaces. Does it mean that every d-dimensional CA is conjugate to a one-dimensional CA?*

Exercise 3 (Subshifts of finite type)**.** *Let* $\Sigma_L \subseteq S^{\mathbb{Z}^d}$ *be a subshift of finite type define by some fine set* L *of finite patterns in* $S^{\mathbb{Z}^d}$*. Show that* Σ_L *is closed and invariant under every shift map.*

Exercise 4 (Subsystems and factors of non-surjective CAs)**.** *Let* $\mathcal{A} = (S^{\mathbb{Z}^d}, F)$ *be a non-surjective CA.*

a) Construct a SFT $\Sigma \subset S^{\mathbb{Z}^d}$ *which yields a subsystem of* $\mathcal{A}$*.*

b) Let $\mathcal{O} = \{\mathbf{2}^{\mathbb{Z}^d}, O\}$ *be the constant CA mapping every configuration to* $0^{\mathbb{Z}^d}$*. Show that* $\mathcal{O}$ *is a factor of* $\mathcal{A}$*.*

Exercise 5 (Strong conjugacies, embeddings and factor maps)**.** *Let* $\mathcal{A} = (S^{\mathbb{Z}^d}, F)$ *be an arbitrary dynamical system and let* $\mathcal{B} = (T^{\mathbb{Z}^d}, G)$ *be a cellular automaton. Prove the following:*

(i) If $\mathcal{A}$ *and* $\mathcal{B}$ *are conjugate via* $h : S^{\mathbb{Z}^d} \to T^{\mathbb{Z}^d}$ *which commutes with all shifts, then* $\mathcal{A}$ *is a cellular automaton.*

(ii) If $\mathcal{A}$ *embeds into* $\mathcal{B}$ *via a continuous injection* $\iota : S^{\mathbb{Z}^d} \hookrightarrow T^{\mathbb{Z}^d}$ *which commutes with all shifts, then* $\mathcal{A}$ *is a cellular automaton.*

(iii) If $\mathcal{A}$ *is a factor of* $\mathcal{B}$ *via a continuous surjection* $\pi : T^{\mathbb{Z}^d} \twoheadrightarrow S^{\mathbb{Z}^d}$ *which commutes with all shifts, then* $\mathcal{A}$ *is a cellular automaton.*

Exercise 6 (Fixed points under conjugacies, factors, and embeddings)**.** *Let* $\mathcal{A} = (X, F)$ *and* $\mathcal{B} = (Y, G)$ *be topological dynamical systems. What is the relationship between* $\mathrm{Fix}(\mathcal{A}) = \{x \in X \mid F(x) = x.\}$ *and* $\mathrm{Fix}(\mathcal{B}) = \{y \in Y \mid G(y) = y.\}$ *given that:*

(i) $\mathcal{A}$ *and* $\mathcal{B}$ *are conjugate* *(ii)* $\mathcal{A}$ *embeds into* $\mathcal{B}$ *(iii)* $\mathcal{A}$ *is a factor of* $\mathcal{B}$

Exercise 7 (Composing embeddings and factors)**.** *Let* $\mathcal{A} = (X, F), \mathcal{B} = (Y, G)$*, and* $\mathcal{C} = (Z, H)$ *be dynamical systems with their underlying spaces being compact metric spaces. Prove the following:*

(i) *If $\mathcal{A}$ embeds into $\mathcal{B}$ and $\mathcal{B}$ embeds into $\mathcal{C}$ then $\mathcal{A}$ embeds into $\mathcal{C}$.*

(ii) *If $\mathcal{A}$ is a factor of $\mathcal{B}$ and $\mathcal{B}$ is a factor of $\mathcal{C}$ then $\mathcal{A}$ is a factor of $\mathcal{C}$.*

(iii) *If $\mathcal{A}$ embeds into $\mathcal{B}$ and $\mathcal{B}$ is a factor of $\mathcal{C}$ then $\mathcal{A}$ is a factor of some system which embeds into $\mathcal{C}$.*